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LETTER TO THE EDITOR

New deformed Heisenberg oscillator

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Abstract. The discrete spectrum of a deformed oscillator is calculated here for the first time according to the non-canonical Heisenberg algebra. The spectrum extends from $\hbar\omega/2(1+\mu)$ to $\hbar\omega/\mu$, where μ is a positive deformation parameter.

It is well known that the usual (canonical) commutation relations between the position and momentum operators, x and p, were introduced by Heisenberg. It is less well known, however, that, three decades ago, Heisenberg himself proposed generalizing the commutation rules to a non-canonical form [1]. This new idea by Heisenberg was subsequently developed by some authors [2–5] and applied to various physical problems.

A specific form of non-canonical commutation relation for x and p has the following form

$$[x, p] = \mathrm{i}\hbar f(H) \tag{1}$$

where $f(H)$ is an arbitrary Hermitian function of the Hamiltonian H.

Recently, Janussis [6] proved that the deformation Q-Lie algebra is a particular case of a Lie-admissible algebra. The time evolution of the operators of the Q-oscillators was derived for the first time by exploiting the connection between the Q-deformation algebra and the Lie-admissible algebras [7].

According to Santilli [8], the new commutation rules for the Lie-admissible algebras have the following form:

$$xTp - pRx = \mathrm{i}\hbar\Omega(x, p, t, \ldots) \tag{2}$$

where T, R are suitable operators (supposed to present, in general, non-conservative interactions) and $\Omega(x, p, t, \ldots)$ is the operational form of the Lie-admissible tensor.

It can be seen from the above generalization that Heisenberg's non-canonical commutation relation (1) is a particular case of (2) for $T = R = 1$ and $\Omega(x, p, t, \ldots) = f(H)$. Also the connection between quantum groups and Lie-admissible Q-algebras has been extensively studied in [6]. Moreover, Janusis and collaborators introduced the generalized commutation relation [6, 7]

$$(A, A^+) = AA^+ - A^+QA = F(\hat{n}) \tag{3}$$

where $\hat{n}$ is the usual number operator $|\hat{n}|n\rangle = n|n\rangle$ satisfying the commutation rules

$$[A, \hat{n}] = A \qquad [A^+, \hat{n}] = -A^+ \tag{4}$$

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and $F(\hat{n})$ is a suitable function. In the same way we can study the Lie-admissible commutation relation

$$(A, A^+) = AT(\hat{n})A^+ - A^+R(\hat{n})A = 1 \tag{5}$$

where the operators $T(\hat{n})$ and $R(\hat{n})$ are Hermitian. In the present letter we study the cases

$$T(\hat{n}) = f(\hat{n}+1) \qquad R(\hat{n}) = Qf(\hat{n}+2) \tag{6}$$

where $f(\hat{n}+1)$ is a suitable function and $(Q \in [-1, +\infty], Q \neq 0)$.

We apply the bosonization method [5, 9] and seek A, A^+ in the form

$$A = G(\hat{n}+1)a \qquad A^+ = a^+G(\hat{n}+1) \tag{7}$$

and a, a^+, $a^+a = \hat{n}$ are boson operators satisfying the usual commutation relations. Then, from (6) we get for the operator $G(\hat{n})$

$$(\hat{n}+1)f(\hat{n}+2)G^2(\hat{n}+1) - Q\hat{n}f(\hat{n}+1)G^2(\hat{n}) = 1. \tag{8}$$

The solution of the above equation has the form

$$G(\hat{n}+1) = \sqrt{\frac{[\hat{n}+1]}{(\hat{n}+1)f(\hat{n}+2)}} \tag{9}$$

and the relations (7) yield

$$A = \sqrt{\frac{[\hat{n}+1]}{(\hat{n}+1)f(\hat{n}+2)}}\, a \qquad A^+ = a^+\sqrt{\frac{[\hat{n}+1]}{(\hat{n}+1)f(\hat{n}+2)}} \tag{10}$$

where

$$[K] = \frac{Q^K - 1}{Q - 1}.$$

From the above relations we obtain

$$AA^+ = \frac{[\hat{n}+1]}{f(\hat{n}+2)} \qquad A^+A = \frac{[\hat{n}]}{f(\hat{n}+1)} \tag{11}$$

$$[A, A^+] = \frac{[\hat{n}+1]}{f(\hat{n}+2)} - \frac{[\hat{n}]}{f(\hat{n}+1)} \tag{12}$$

$$\{A, A^+\} = AA^+ + A^+A = \frac{[\hat{n}+1]}{f(\hat{n}+2)} + \frac{[\hat{n}]}{f(\hat{n}+1)}. \tag{12a}$$

According to Jannussis *et al* [5], the Fock representation of the operators x and p in the forms of A, A^+ reads in this case

$$x = \frac{1}{2}\sqrt{\frac{\hbar(Q+1)}{m\omega}}\,(A + A^+) \tag{13}$$

$$p = -\frac{\mathrm{i}}{2}\sqrt{\hbar m\omega(Q+1)}\,(A - A^+) \tag{14}$$

and the Hamilton operator of the harmonic oscillator takes the form

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2x^2 = \frac{\hbar\omega(Q+1)}{4}(AA^+ + A^+A). \tag{15}$$

Furthermore, the operators A^+A and $\hat{n}$ commute, i.e.

$$[A^+A, \hat{n}] = 0 \tag{16}$$

and have the common basis $|n\rangle$, $\langle l|n\rangle=\delta_{ln}$ of bosons. Acting on $|n\rangle$ with A and A^+ gives

$$A|n\rangle=\sqrt{\frac{[n]}{f(n+1)}}|n-1\rangle \tag{17}$$

$$A^+|n\rangle=\sqrt{\frac{[n+1]}{f(n+2)}}|n+1\rangle \tag{18}$$

$$A^+A|n\rangle=\frac{[n]}{f(n+1)}|n\rangle \tag{19}$$

and for $f(K)\neq 0$, $K=1,2,\ldots$ the operators A and A^+ are exactly annihilation and creation operators. From (12a) and (15) we obtain the eigenvalues of the Hamiltonian operator H, i.e.

$$H|n\rangle=\frac{\hbar\omega(Q+1)}{4}\left(\frac{[n+1]}{f(n+2)}+\frac{[n]}{f(n+1)}\right)|n\rangle \tag{20}$$

which in the case $f(\hat{n}+1)=1$ reduces to the spectrum of the Q-harmonic oscillator [5, 9]. Also from (20) we obtain for $Q=1$ the eigenvalues

$$H|n\rangle=\frac{\hbar\omega}{2}\left(\frac{n+1}{f(n+2)}+\frac{n}{f(n+1)}\right)|n\rangle. \tag{21}$$

An interesting case is $f(n+1)=1+\mu n$, $\mu>0$, for which the eigenvalues take the form

$$H|n\rangle=\frac{\hbar\omega}{2}\left(\frac{n+1}{1+\mu(n+1)}+\frac{n}{1+\mu n}\right)|n\rangle=E_n|n\rangle. \tag{22}$$

In the following, we will call the above oscillator the 'deformed Heisenberg oscillator', since its discrete spectrum extends from the ground energy $E_0=\hbar\omega/2(1+\mu)$ to the upper limit energy $E_\infty=\hbar\omega/\mu$.

To our knowledge, this is the first time where an oscillator spectrum bounded from above is presented. The nature of the above spectrum is a direct consequence of the introduction of the parameter μ in the non-canonical commutation relation of the deformed Heisenberg algebra. The operator

$$H=\frac{\hbar\omega}{2}\left(\frac{\hat{n}+1}{1+\mu(\hat{n}+1)}+\frac{\hat{n}}{1+\mu\hat{n}}\right) \tag{23}$$

takes the following form (for $\hat{n}=H_0/\hbar\omega-\frac{1}{2}$):

$$H=\frac{\hbar\omega}{2}\left(\frac{H_0/\hbar\omega+\frac{1}{2}}{1+\mu(H_0/\hbar\omega+\frac{1}{2})}+\frac{H_0/\hbar\omega-\frac{1}{2}}{1+\mu(H_0/\hbar\omega-\frac{1}{2})}\right) \tag{24}$$

where $H_0=\hbar\omega(\hat{n}+\frac{1}{2})$ is the Hamiltonian of the usual harmonic oscillator.

After some algebra the commutator of the operators x and p can be written

$$[x,p]=\mathrm{i}\hbar\frac{2(1-\mu H/\hbar\omega)^2}{1+\sqrt{1+\mu^2(1-\mu H/\hbar\omega)^2}} \tag{25}$$

which is a particular case of the non-canonical commutation relation (1). The above commutation for $\mu=0$ reduces exactly to the usual canonical form.

Before constructing the corresponding quantum group we generalize the commutation relation (25) to n dimensions, i.e.

$$A_j(1+\mu_j\delta_{jk}\hat{n}_j)A_k^+ - A_k^+(1+\mu_k\delta_{jk}(\hat{n}_k+1))A_j = \delta_{jk}. \tag{26}$$

From the above relations we obtain

$$A_jA_k^+ = A_k^+A_j \qquad \text{for } i\neq k \tag{27}$$

$$A_j(1+\mu_j\hat{n}_j)A_j^+ - A_j^+(1+\mu_j(\hat{n}_j+1))A_j = 1 \tag{28}$$

$$A_jA_k = A_kA_j \qquad A_j^+A_k^+ = A_k^+A_j^+ \tag{29}$$

and from (28) we have

$$A_j = \frac{1}{\sqrt{1+\mu_j(\hat{n}_j+1)}}a_j \qquad A_j^+ = a_j^+\frac{1}{\sqrt{1+\mu_j(\hat{n}_j+1)}} \tag{30}$$

$$A_j|n_1, n_2, \ldots, n_j, \ldots\rangle \sqrt{\frac{n_j}{1+\mu_jn_j}}|n_1, n_2, \ldots, n_j-1, \ldots\rangle \tag{31}$$

$$A_j^+|n_1, n_2, \ldots, n_j, \ldots\rangle = \sqrt{\frac{n_j+1}{1+\mu_j(n_j+1)}}|n_1, n_2, \ldots, n_j+1, \ldots\rangle \tag{32}$$

$$A_j^+A_j|n_1, n_2, \ldots, n_j, \ldots\rangle = \frac{n_j}{1+\mu_jn_j}|n_1, n_2, \ldots, n_j, \ldots\rangle. \tag{33}$$

By using the definition

$$J_+ = A_1^+A_2 \qquad J_- = A_2^+A_1 \qquad [J_+, J_-] = 2J_z \tag{34}$$

it is easy to construct the quantum group.

Substituting the expressions (30) in (34) we obtain

$$J_+ = \frac{1}{\sqrt{(1+\mu_1\hat{n}_1)(1+\mu_2(\hat{n}_2+1))}}a_1^+a_2 \tag{35}$$

$$J_- = \frac{1}{\sqrt{(1+\mu_1(\hat{n}_1+1))(1+\mu_2\hat{n}_2)}}a_2^+a_1 \tag{36}$$

$$2J_z = \frac{(\hat{n}_1-\hat{n}_2)+\mu_1\hat{n}_1(\hat{n}_1+1)-\mu_2\hat{n}_2(\hat{n}_2+1)}{(1+\mu_1\hat{n}_1)(1+\mu_1(\hat{n}_1+1))(1+\mu_2\hat{n}_2)(1+\mu_2(\hat{n}_2+1))}. \tag{37}$$

Furthermore, with $\hat{n}_j|n_1n_2\rangle = n_j|n_1n_2\rangle$, $j=1,2$, we have

$$J_z|n_1, n_2\rangle = \frac{1}{2}\frac{(n_1-n_2)+\mu_1n_1(n_1+1)-\mu_2n_2(n_2+1)}{(1+\mu_1n_1)(1+\mu_1(n_1+1))(1+\mu_2\hat{n}_2)(1+\mu_2(\hat{n}_2+1))}|n_1, n_2\rangle \tag{38}$$

$$J_+|n_1, n_2\rangle = \frac{\sqrt{n_2(n_1+1)}}{\sqrt{(1+\mu_2n_2)(1+\mu_1(\hat{n}_1+1))}}|n_1+1, n_2-1\rangle \tag{39}$$

$$J_-|n_1, n_2\rangle = \frac{\sqrt{n_1(n_2+1)}}{\sqrt{(1+\mu_1n_1)(1+\mu_2(\hat{n}_2+1))}}|n_1-1, n_2+1\rangle. \tag{40}$$

Similarly, we can construct the quantum group for any function $f(\hat{n}_1+2)$.

It is hoped that the new deformed oscillator will find applications in physics.

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